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**DIRICHLET AVERAGE OF NEW GENERALIZED M-SERIES AND FRACTIONAL  
DERIVATIVE**

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**ABSTRACT**

In this note we set up a relation between Dirichlet average of New Generalized M-series, and fractional derivative.

**KEYWORDS AND PHRASES:** Dirichlet average New Generalized M-series, fractional derivative and Fractional calculus operators.

**Mathematics Subject Classification:** 26A33, 33A30, 33A25 and 83C99.

**INTRODUCTION**

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions like  $x^t, e^x$  etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging  $x^n, e^x$  etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

In this paper the Dirichlet average of **New Generalized M-series** has been obtained.

**DEFINITIONS**

We give below some of the definitions which are necessary in the preparation of this paper.

**Standard Simplex in  $R^n, n \geq 1$ :**

We denote the standard simplex in  $R^n, n \geq 1$  by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \dots, u_n) : u_1 \geq 0, \dots, u_n \geq 0, u_1 + u_2 + \dots + u_n \leq 1\} \quad (2.1.1)$$

**Dirichlet measure:**

Let  $b \in C^k, k \geq 2$  and let  $E = E_{k-1}$  be the standard simplex in  $R^{k-1}$ . The complex measure  $\mu_b$  is defined by  $E[1]$ .

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \dots - u_{k-1})^{b_k-1} du_1 \dots du_{k-1} \quad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b_1, \dots, b_k) = \frac{\Gamma(b_1) \dots \Gamma(b_k)}{\Gamma(b_1 + \dots + b_k)},$$

$$C_{>} = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \pi/2\},$$

Open right half plane and  $C_{>}^k$  is the  $k^{\text{th}}$  Cartesian power of  $C_{>}$

**Dirichlet Average[1, p.75]:**

Let  $\Omega$  be the convex set in  $C_{>}$ , let  $z = (z_1, \dots, z_k) \in \Omega^k, k \geq 2$  and let  $u.z$  be a convex combination of  $z_1, \dots, z_k$ . Let  $f$  be a measurable function on  $\Omega$  and let  $\mu_b$  be a Dirichlet measure on the standard simplex  $E$  in  $R^{k-1}$ . Define

$$F(b, z) = \int_E f(u, z) d\mu_b(u) \tag{2.3.1}$$

We shall call F the Dirichlet measure of  $f$  with variables  $z = (z_1, \dots, z_k)$  and parameters  $b = (b_1, \dots, b_k)$ . Here

$$u, z = \sum_{i=1}^k u_i z_i \text{ and } u_k = 1 - u_1 - \dots - u_{k-1} \tag{2.3.2}$$

If  $k = 1$ , define  $F(b, z) = f(z)$ .

**Fractional Derivative [8, p.181]:**

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order  $\alpha$  found in the literature on the ‘‘Riemann-Liouville integral’’ is

$$D_z^\alpha F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.4.1}$$

Where  $Re(\alpha) < 0$  and  $F(x)$  is the form of  $x^p f(x)$ , where  $f(x)$  is analytic at  $x = 0$ .

**THE NEW GENERALIZED M-SERIES**

The New Generalized M-series is defined as follows.

$${}_pM_q^{\alpha, \beta}(a_1 \dots a_p; b_1 \dots b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^{(k+\frac{\beta-1}{\alpha})}}{\Gamma(\alpha k + \beta)} \tag{3.1}$$

Here  $\alpha, \beta \in C, R(\alpha) > 0, R(\beta) > 0; (a_j)_k, (b_j)_k$  are pochhammer symbols.

**Special Case 1: If There is no upper and lower parameter in equation (3.1), It converts in Agarwal’s Function**

The Agarwal’s Function is a generalization of Mittag-Leffler function given by Agarwal (1953) as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^{(k+\frac{\beta-1}{\alpha})}}{\Gamma(\alpha k + \beta)} \tag{3.2}$$

**Special Case 2 : If There is  $\beta = 1$  in equation (3.1) It converts in M-Series**

**The M-series:**

The M-series is a particular case of the well known H- function . It plays a special role is in the application of fractional calculus operators and in the solutions of fractional order differential equations. The Hypergeometric function and Mittag-Laffler function follow as its particular case. Therefore, it is very interesting. The M-series was introduced by Sharma [291]:

$${}_pM_q^\alpha(a_1 \dots a_p; b_1 \dots b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{3.3}$$

Here,  $\alpha, \beta \in C, R(\alpha) > 0, (a_j)_k (b_j)_k$  are pochhammer symbols.

**EQUIVALENCE**

In this section we shall show the equivalence of single Dirichlet average of New

**Generalized M-series**( $k = 2$ ) with the fractional derivative i.e.

$$S(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma\beta} (x - y)^{1-\beta-\beta'} D_{x-y}^{-\beta'} M_q^{\alpha, \beta}(x)(x - y)^{\beta-1} \tag{4.1}$$

**Proof:**

$$S(\beta, \beta'; x, y) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^{\left(\frac{\beta-1}{\alpha}\right)}}{\Gamma(\alpha k + \beta)} R_n(\beta, \beta'; x, y)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{1}{\Gamma(\alpha k + \beta)} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^1 [ux + (1-u)y]^{(n+\frac{\beta-1}{\alpha})} u^{\beta-1} (1-u)^{\beta'-1} du$$

Putting  $u(x-y) = t$ , we have,

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{\Gamma(\beta + \beta')}{\Gamma(\alpha n + \beta) \Gamma\beta \Gamma\beta'} \int_0^{x-y} [t+y]^{(n+\frac{\beta-1}{\alpha})} \left(\frac{t}{x-y}\right)^{\beta-1} \left(1-\frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^{x-y} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{1}{\Gamma(\alpha n + \beta)} [t+y]^{(n+\frac{\beta-1}{\alpha})} (t)^{\beta-1} (x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} \int_0^{x-y} {}_pM_q^{\alpha, \beta}(x) (t)^{\beta-1} (x-y-t)^{\beta'-1} dt$$

Hence by the definition of fractional derivative, we get

$$S(\beta, \beta'; x, y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta + \beta')}{\Gamma\beta} D_{x-y}^{-\beta'} {}_pM_q^{\alpha, \beta}(x) (x-y)^{\beta-1}$$

This completes the Analysis.

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